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Exact solutions to the forced KdV equation via three efficient techniques

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Abstract. In this work, the exact travelling wave solutions to the forced Korteweg–de Vries (fKdV) equation with different force terms are studied with the help of symbolic computations. This equation is derived from a straightforward mathematical model that describes the behaviour of a shallow fluid layer when influenced by external forces. The fKdV equation has many applications in diverse fields, including fluid dynamics, plasma physics, soliton theory and mathematical physics, for modeling wave propagation and nonlinear phenomena under the influence of external forces. Solitary wave solutions for this equation have been derived using three distinct techniques: the extended (G'/G)-expansion method, the Kudryashov method and the $(1/G')$-expansion method. As a result, several new solutions have been achieved which are in the form of hyperbolic, trigonometric, rational and exponential functions. Finally, the effects of different time-dependent external forces have been studied by presenting 3D, 2D and contour plots. It can be seen that the external forces affect the background and speed of solitary waves. The results could be expected to be helpful in understanding the propagation of solitary waves subjected to external forces.

Keywords. Extended (G'/G)-expansion method; Kudryashov method; $(1/G')$-expansion method; Korteweg–de Vries equation with forcing term; Solitary waves.

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1. Introduction

Nonlinear evolution equations (NLEEs) can simulate complex natural events, which have many mathematical applications. In line with recent advances in studying and modelling complicated physical phenomena, more and more NLEEs have been used with constant and variable coefficients. A class of higher-dimensional NLEEs is found in invariance analysis and solitary wave theory. Such types of equations are the Bogoyvlenskip Scheff equation [1], deoxyribonucleic acid model [2], Konopelchenko Dubrovsky system [3], coupled Davey Stewartson Fokas system [4], $(3+1)$-dimensional KdV Benjamin Bona Mahony equation [5] and so on.

There is a great interest in finding exact solutions to NLEEs because they provide insight into the nonlinear phenomena and numerous elements of physical phenomena. Using high-performance computers, efficient approaches and trustworthy algorithms are devised to study exact solutions. The mission to find analytical solutions of NLEEs is crucial for understanding the most extreme nonlinear events. From the perspective of practical applications in physics, there is a widespread requirement for exact solutions to NLEEs to assess the effectiveness of approximations and computational methods. Many analytical methods, such as the exp-function method [6,7], the Lie symmetry technique [8], the sine–cosine method [9], the modified simple equation technique [10,11], the Hirota's bilinear method [12], the variational iteration method [13], the He's homotopy perturbation technique [14], the (G'/G) expansion technique and its several variations [15–22], the Adomian decomposition method [23], the generalised exponential rational function technique [24], the tanh-function procedure [25], the generalised sub-ODE approach [26], the improved tanh process [27], the modified Kudryashov process [28–30], etc. for analysing the exact solutions of the NLEEs have been devised.

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Out of these methods, the extended (G'/G) expansion method, the Kudryashov method and the $(1/G')$ expansion method are straightforward methods. In these three methods, the first step is to convert the NLEEs into ordinary differential equations (ODEs) using a wave transformation, with the procedures being similar but differing in their initial assumptions. In the extended (G'/G)-expansion method [21], the initial assumption solutions for the ODE are of the form $\sum_{i=-n}^{n} j_i(G'/G)^i$, where $G = G(\zeta)$ satisfies the ODE of the form $G'' + \nu G' + \mu G = 0$ in which ν and μ are constants. In the Kudryashov method [28], the initial assumption solutions for the ODE are of the form $\sum_{i=0}^{m} d_i(Z(\zeta))^i$, where $Z = Z(\zeta)$ satisfies the ODE of the form $Z' = Z^2 - Z$. In the $(1/G')$-expansion method [31], the initial assumption solutions are of the form $\sum_{i=0}^{n} \alpha_i(1/G'(\zeta))^i$ where α_i, $i = 1, 2, \ldots, n$, are constants and $G(\zeta)$ satisfies the ODE, $G'' + P_1 G' + P_2 = 0$, in which P_1 and P_2 are constants.

Pelinovsky *et al* [32] presented an analytical model for the propagation of tsunamis in the following manner:

$$\eta_t + c\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = F_x, \tag{1}$$

with

$$\alpha = \frac{3c}{2h_0}, \quad \beta = \frac{ch_0^2}{6}, \quad F = -\frac{cz}{2}. \tag{2}$$

$\eta = \eta(x, t)$ represents the height of the free water surface, while $z = z(x, t)$ denotes the position of the solid bottom. The constant mean water depth is represented by h and c is approximately equal to the square root of the product of gravity acceleration g and h, representing the speed of long waves. If $F_x = 0$, indicated on the right-hand side of eq. (1), the equation transforms into the KdV equation, which is fully integrable [33]. However, when $F_x \neq 0$, the equation becomes challenging to be integrable due to the presence of the forcing term, known as F_x. Consequently, (1) is referred to as the 'KdV equation with a force term' or the 'forced KdV (fKdV) equation' [34,35]. Since Pelinovsky's groundbreaking work, the development of explicit asymptotic derivations [36,37] for the fKdV equation has garnered much more attention. In the recent past, a particular kind of fKdV equation using Hirota's bilinear technique was examined in [35,38]. Additionally, conservation laws and characteristics of integrability for a special fKdV was studied in [39,40]. The fKdV equation is extensively used to describe the propagation of waves with tiny amplitudes generated by moving sources with constant speed and small amplitudes [41,42]. This model, for example, has been applied to study flows through obstacles in hydrodynamics, ship waves, trapped waves [43,44], internal waves in stratified fluid flows [45] and many others. The fKdV equation is also used to explain nonlinear wave excitations resulting from the orbital motion of a charged space debris object in the plasma environment of the low Earth orbit region [46], nonlinear waves in plasmas [47], superthermal plasmas [48] and so on. In general, the fKdV equation is a standard model for explaining the interaction of a soliton with an external force in a variety of physical media.

The form of fKdV equation that we study is as follows [49]:

$$u_t + \alpha u u_x + \beta u_{xxx} = f(t), \tag{3}$$

where α and β denote the non-linear coefficient and dispersion coefficient, respectively, while $f(t)$ represents an external force. When $f(t) = 0$, eq. (3) reduces to KdV equation with constant coefficients. The KdV equation has been used to describe long wavelength waves at the fluid surface, waves in crystals, ion-acoustic waves, magnetohydrodynamic waves in plasma and so on. The theory of solitons associated with the KdV equation is well established at present due to the existence of analytical tools. However, the results are based on the analysis of homogeneous equations. Since wave motion in complicated media may be influenced by the additional energy influx caused by the accompanying processes, it is necessary to consider an external force in the original model [50,51]. For example, in geophysical and marine scenarios, the consideration of an external force becomes crucial, especially when waves are generated by factors such as shear instabilities [50], seismic events [52], the motion of ships, as well as flows over bottom topography [53,54]. Again, in the case of acoustic waves in plasma, such an external force (noise) may originate from the laser radiation [55]. It is, therefore, interesting to study the nonlinear wave motion modelled by the KdV equation in a more general setting, including the presence of various external fields. It should be emphasised that the external force term $f(t)$ in eq. (3) should be a function of both space and time for it to have finite energy and spatial domain in general. In cases where the spatial region containing solitary waves is much smaller than the force domain, it may be possible to consider the force term dependent on time alone [56,57]. Even if model (3) is more realistic when the force term depends on both position and time, it is much harder to investigate using analytical techniques. One advantage of our simple model is its tractability through analytical methods, enabling us to obtain exact results. The choice of the force term with time only can also be seen in [51,56–58] and the references therein. In our current investigation, we examine three distinct types of forces: the periodic force $(f_0 \sin(\omega t))$ [59,60], the Gaussian-shape force $(g_0 \exp(-\lambda^2 t^2))$ [57,61] and the hyperbolic force $(h_0 \text{sech}^2(\delta t))$ [35,61]. The parameters f_0, g_0 and h_0 govern the amplitude, while ω, λ and δ

determine the width of these forces. Note that, eq. (3) can be obtained by substituting

$$u = \frac{c}{\alpha} + \eta$$

and

$$F = xf(t)$$

into (1). Zhang [49] studied eq. (3) using the exp-function method to obtain the exact solutions. Manafian and Nasrollahpour [62] investigated the same equation using (G'/G)-expansion approach and generalised tanh–coth process to establish analytical solutions. To extract exact solutions of the Jacobi elliptic function, the same equation has been solved by Tao [63] using the process of auxiliary equation with the transformation of a function. Additionally, Zun-tao *et al* [64] explored this equation via extended Jacobi elliptic function expansion technique.

To the best of our assessment, the fKdV equation of the form (3) has not been explored by the extended (G'/G)-expansion approach [21], the Kudryashov method [28] and the $(1/G')$-expansion technique [31]. Building on the motivations outlined above, this paper aims to employ the previously mentioned three methods to obtain exact wave solutions for eq. (3). Additionally, the objective is to elucidate how various forces affect the resulting solitary waves.

The structure of the paper is as follows: Section 2 outlines the methods, §3 applies the extended (G'/G)-expansion method, the Kudryashov method and the $(1/G')$-expansion method to investigate the solutions of fKdV equation, §4 presents the results, discussion and comparison of the study and §5 gives the conclusion of this work.

2. Explanation of the methods

Consider an NLEE as

$$M(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{4}$$

where $u = u(x,t)$ is the unidentified function. By introducing the reformed wave variable $u = W(\zeta)$, where $\zeta = \zeta(x,t)$ exhibits linear dependence on both x and t, the equation can be transformed into an ordinary ODE,

$$N(W, W', W'', W''', \ldots) = 0, \tag{5}$$

where the prime denotes the differentiation with respect to ζ.

In order to discover solutions, we can perform the integration of (5) once or multiple times and assign a value of zero to the unidentified constant of integration. The following subsections outline the fundamental steps involved in the proposed approaches.

2.1 *The extended (G'/G)-expansion method* [21]

The key steps of this method are as follows:

Step 1: According to the extended (G'/G)-expansion technique, the solution of eq. (5) is taken as follows:

$$W(\zeta) = j_0 + \sum_{i=1}^{l} j_i (G'/G)^i + k_i (G'/G)^{-i}, \tag{6}$$

where the constants j_0, j_i, k_i $(i = 1, 2, \ldots, l)$ will be determined at a later stage and $G = G(\zeta)$ satisfies the following ODE:

$$G'' + \nu G' + \mu G = 0, \tag{7}$$

where ν and μ are constants. The general solutions of eq. (7) are given below in different conditions,

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\nu}{2} + \frac{\sqrt{\nu^2-4\mu}}{2} \times \frac{P\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right) + Q\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}{P\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right) + Q\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}, \\ \text{when} \quad \nu^2-4\mu > 0, \\ -\frac{\nu}{2} + \frac{\sqrt{4\mu-\nu^2}}{2} \times \frac{P\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right) - Q\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}{P\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right) + Q\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}, \\ \text{when} \quad \nu^2-4\mu < 0, \\ -\frac{\nu}{2} + \frac{Q}{P+Q\zeta}, \quad \text{when} \quad \nu^2-4\mu = 0, \end{cases} \tag{8}$$

where P and Q are arbitrary constants.

Step 2: In order to determine the positive integer l in eq. (6), it is required to balance the exponent between highest-order derivative terms and the highest order non-linear terms in eq. (5).

Step 3: Substitute the value of l found in Step 2 and eqs (6) and (7) into eq. (5). Equation (5) becomes a polynomial of (G'/G), then equating the coefficient terms of the same power of (G'/G) to zero will yield a class of algebraic equations. Then, solve the system of algebraic equations for j_0, j_i, k_i and any other necessary constraints.

Step 4: By plugging the solutions provided in eq. (8) and the estimations of j_0, j_i and k_i into solution (6), we can derive comprehensive analytical solutions for eq. (4).

2.2 *The Kudryashov method* [28]

The steps of the Kudryashov method are given as follows for solving eq. (4):

Step 1: First convert the PDE into ODE by using the wave transformation, then, assume the initial approximation solutions for eq. (5) as follows:

$$W(\zeta) = d_0 + \sum_{i=1}^{m} d_i (Z(\zeta))^i, \tag{9}$$

where $d_i, i = 1, 2, \ldots, m$ are unknown constants, $d_m \neq 0$ and $Z(\zeta)$ satisfies the following Riccati equation:

$$Z' = Z^2 - Z. \tag{10}$$

The general solution of eq. (10) is

$$Z(\zeta) = \frac{1}{1 + e^{\zeta}}. \tag{11}$$

Step 2: To determine the value of m in eq. (9), balance the highest-order nonlinear terms with the highest-order derivative in eq. (5).

Step 3: With the value of m and substituting the values of (9) and (10) in eq. (5), it becomes a polynomial of Z. Then, the system of algebraic equations is obtained by equating the coefficients of same power of Z to zero. Further, solve the system for d_0, d_i and any other necessary constraints.

Step 4: We can obtain complete analytical solutions for eq. (4) by inserting the solution obtained from eq. (11) and the obtained values for d_0 and d_i into solution (9).

2.3 *The $(1/G')$-expansion method* [31]

The $(1/G')$-expansion method consists of the following steps:

Step 1: First convert the PDE into ODE by using the wave transformation, then the initial approximation solutions for the ODE is expressed in terms of a polynomial of $(1/G')$ as follows:

$$W(\zeta) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \left(\frac{1}{G'(\zeta)} \right)^i, \tag{12}$$

where α_i, $i = 1, 2, \ldots, n$ are constants, $\alpha_n \neq 0$ and $G(\zeta)$ satisfies the following ODE:

$$G'' + P_1 G' + P_2 = 0, \tag{13}$$

where P_1 and P_2 are constants. The solution of eq. (13) is

$$G(\zeta) = -(P_2 \zeta)/P_1 + s_1 e^{-P_1 \zeta} + s_2, \tag{14}$$

where s_1 and s_2 are constants. Then

$$\frac{1}{G'(\zeta)} = \frac{P_1}{-P_2 + P_1 s_1 (\cosh(\zeta P_1) - \sinh(\zeta P_1))} \tag{15}$$

can be written.

Step 2: To get the value of m in eq. (12), non-linear terms and the the highest-order derivative terms in eq. (5) should be balanced.

Step 3: Substituting the value of eq. (12) into eq. (5) and utilising the second-order ODE (13), the expression on the left side of (5) can be converted into a polynomial represented in relation to $(1/G')$.

Step 4: Equating the coefficients of the same power of $(1/G')$ to zero, a set of algebraic equations is generated and then solve the system for α_0 and α_i. Then substitute the values of α_0 and α_i in eq. (12) and use the value of eq. (15), to get solution for (5).

3. Implementations of the methods

In this section, we apply all the aforementioned methods to extract the solutions of eq. (3). We assume that

$$u(x, t) = w(x, t) + \int f(t) \mathrm{d}t. \tag{16}$$

Subsequently, eq. (3) simplifies to

$$w_t + \alpha w w_x + \alpha \int f(t)\, \mathrm{d}t\, w_x + \beta w_{xxx} = 0. \tag{17}$$

Now, using the wave transformation

$$w(x, t) = W(\zeta), \quad \zeta = cx + \int \rho(t) \mathrm{d}t, \tag{18}$$

where c is a constant and $\rho(t)$ function of t, eq. (17) reduces to the ODE as

$$\rho(t) W' + c\alpha W W' + c\alpha \int f(t) \mathrm{d}t\, W' + c^3 \beta W''' = 0. \tag{19}$$

3.1 *Implementation of the extended (G'/G)-expansion method*

Here, we implement the extended (G'/G)-expansion method to obtain the solutions of eq. (3). Equation (19) yields $l = 2$, if we take the homogeneous balance of (WW') and (W'''). Hence, the form of solution (6) is as follows:

$$W(\zeta) = j_0 + j_1 \left(\frac{G'}{G} \right) + j_2 (\frac{G'}{G})^2 + k_1 \left(\frac{G'}{G} \right)^{-1} + k_2 \left(\frac{G'}{G} \right)^{-2}. \tag{20}$$

By inserting eqs (20) and (7) into (19) and gathering the coefficients for $(G'/G)^i$ and $(G'/G)^{-i}$ (where $i = 0, 1, 2, 3, 4, 5$) and equating them to zero, a set of algebraic equations is obtained. Solving the system of equations using Maple, we have

Set 1:

$$j_0 = j_0, \quad j_1 = 0, \quad j_2 = 0,$$
$$k_1 = \frac{-12c^2\beta\mu\nu}{\alpha}, \quad k_2 = \frac{-12c^2\beta\mu^2}{\alpha},$$
$$\rho(t) = -\beta c^3\nu^2 - 8\beta c^3\mu - c\alpha\int f(t)\mathrm{d}t - \alpha c j_0. \tag{21}$$

Set 2:

$$j_0 = j_0, \quad j_1 = \frac{-12c^2\beta\nu}{\alpha}, \quad j_2 = \frac{-12c^2\beta}{\alpha},$$
$$k_1 = 0, \quad k_2 = 0,$$
$$\rho(t) = -\beta c^3\nu^2 - 8\beta c^3\mu - c\alpha\int f(t)\mathrm{d}t - \alpha c j_0. \tag{22}$$

Solutions for Set 1:
Substituting the estimations of the constants from (21) and (22) into eq. (20), we get

$$W_1(\zeta) = j_0 - \frac{12c^2\beta\mu\nu}{\alpha}(G'/G)^{-1} - \frac{12c^2\beta\mu^2}{\alpha}(G'/G)^{-2}. \tag{23}$$

Substituting the different solutions of (8) into (23) and with the help of (16) and (18), we have the following three types of exact wave solutions of eq. (3).

When $\nu^2 - 4\mu > 0$, we get the hyperbolic solution

$$u_1(\zeta) = j_0 - \frac{12c^2\beta\mu\nu}{\alpha}\left(-\frac{\nu}{2} + \frac{\sqrt{\nu^2-4\mu}}{2} \times \frac{P\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right) + Q\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}{P\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right) + Q\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}\right)^{-1}$$
$$- \frac{12c^2\beta\mu^2}{\alpha}\left(-\frac{\nu}{2} + \frac{\sqrt{\nu^2-4\mu}}{2} \times \frac{P\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right) + Q\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}{P\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right) + Q\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}\right)^{-2} + \int f(t)\mathrm{d}t, \tag{24}$$

where

$$\zeta = cx + \int\left(-\beta c^3\nu^2 - 8\beta c^3\mu - c\alpha\int f(t)\mathrm{d}t - \alpha c j_0\right)\mathrm{d}t.$$

The value of ζ is the same for the rest of the solutions in Set 1.

If $P = 0$ and $Q \neq 0$, solution (24) reduces to

$$u_{11}(\zeta) = j_0 - \frac{12c^2\beta\mu\nu}{\alpha}\left(-\frac{\nu}{2} + \frac{\sqrt{\nu^2-4\mu}}{2} \times \tanh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)\right)^{-1} - \frac{12c^2\beta\mu^2}{\alpha} \times \left(-\frac{\nu}{2} + \frac{\sqrt{\nu^2-4\mu}}{2} \times \tanh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)\right)^{-2} + \int f(t)\mathrm{d}t. \tag{25}$$

When $\nu^2 - 4\mu < 0$, we get the trigonometric solution

$$u_2(\zeta) = j_0 - \frac{12c^2\beta\mu\nu}{\alpha}\left(-\frac{\nu}{2} + \frac{\sqrt{4\mu-\nu^2}}{2} \times \frac{P\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right) - Q\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}{P\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right) + Q\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}\right)^{-1}$$
$$- \frac{12c^2\beta\mu^2}{\alpha}\left(-\frac{\nu}{2} + \frac{\sqrt{4\mu-\nu^2}}{2} \times \frac{P\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\xi\right) - Q\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}{P\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right) + Q\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}\right)^{-2} + \int f(t)\mathrm{d}t. \tag{26}$$

If $P = 0$ and $Q \neq 0$, solution (26) gives

$$u_{21}(\zeta) = j_0 - \frac{12c^2\beta\mu\nu}{\alpha}\left(-\frac{\nu}{2} - \frac{\sqrt{4\mu-\nu^2}}{2} \times \tan\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)\right)^{-1} - \frac{12c^2\beta\mu^2}{\alpha}$$

$$-\left(-\frac{\nu}{2}\frac{\sqrt{4\mu-\nu^2}}{2}\times\tan\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)\right)^{-2}+\int f(t)\mathrm{d}t. \quad (27)$$

When $\nu^2-4\mu=0$, we get the rational solution

$$u_3(\zeta)=j_0-\frac{12c^2\beta\mu\nu}{\alpha}\left(-\frac{\nu}{2}+\frac{Q}{P+Q\zeta}\right)^{-1}-\frac{12c^2\beta\mu^2}{\alpha}\left(-\frac{\nu}{2}+\frac{Q}{P+Q\zeta}\right)^{-2}+\int f(t)\mathrm{d}t. \quad (28)$$

If we put $P=0$ and $Q\neq 0$, solution (28) becomes

$$u_{31}(\zeta)=j_0-\frac{12c^2\beta\mu\nu}{\alpha}\left(-\frac{\nu}{2}+\frac{1}{\zeta}\right)^{-1}-\frac{12c^2\beta\mu^2}{\alpha}\times\left(-\frac{\nu}{2}+\frac{1}{\zeta}\right)^{-2}+\int f(t)\,\mathrm{d}t. \quad (29)$$

Solutions for Set 2:

Substituting the estimations of the constants from (22) into eq. (20), we get

$$W_2(\zeta)=j_0-\frac{12c^2\beta\nu}{\alpha}(G'/G)-\frac{12c^2\beta}{\alpha}(G'/G)^2. \quad (30)$$

Similarly, if we use (8) into (30) and take the help of (16) and (18), we get the wave solutions of eq. (3) as follows.

When $\nu^2-4\mu>0$, we obtain the hyperbolic solution

$$u_4(\zeta)=j_0-\frac{12c^2\beta\nu}{\alpha}\left(-\frac{\nu}{2}+\frac{\sqrt{\nu^2-4\mu}}{2}\times\frac{P\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)+Q\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}{P\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)+Q\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}\right)-\frac{12c^2\beta}{\alpha}\left(-\frac{\nu}{2}+\frac{\sqrt{\nu^2-4\mu}}{2}\times\frac{P\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)+Q\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}{P\sinh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)+Q\cosh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)}\right)^2+\int f(t)\mathrm{d}t, \quad (31)$$

where

$$\zeta=cx+\int\left(-\beta c^3\nu^2-8\beta c^3\mu-c\alpha\int f(t)\mathrm{d}t-\alpha c j_0\right)\mathrm{d}t.$$

The value of ζ remains the same for the remaining solutions in Set 2.

If $P=0$ and $Q\neq 0$, after reduction solution (31) reduces to

$$u_{41}(\zeta)=j_0-\frac{12c^2\beta\nu}{\alpha}\left(-\frac{\nu}{2}+\frac{\sqrt{\nu^2-4\mu}}{2}\times\tanh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)\right)-\frac{12c^2\beta}{\alpha}\left(-\frac{\nu}{2}+\times\frac{\sqrt{\nu^2-4\mu}}{2}\times\tanh\left(\frac{\sqrt{\nu^2-4\mu}}{2}\zeta\right)\right)^2+\int f(t)\mathrm{d}t. \quad (32)$$

When $\nu^2-4\mu<0$, we get the trigonometric solution

$$u_5(\zeta)=j_0-\frac{12c^2\beta\nu}{\alpha}\left(-\frac{\nu}{2}+\frac{\sqrt{4\mu-\nu^2}}{2}\times\frac{P\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)-Q\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}{P\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)+Q\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}\right)-\frac{12c^2\beta}{\alpha}\left(-\frac{\nu}{2}+\frac{\sqrt{4\mu-\nu^2}}{2}\times\frac{P\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)-Q\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}{P\sin\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)+Q\cos\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)}\right)^2+\int f(t)\mathrm{d}t. \quad (33)$$

If $P=0$ and $Q\neq 0$, after lightening, solution (33) gives

$$u_{51}(\zeta)=j_0-\frac{12c^2\beta\nu}{\alpha}\left(-\frac{\nu}{2}-\frac{\sqrt{4\mu-\nu^2}}{2}\times\tan\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)\right)-\frac{12c^2\beta}{\alpha}\left(-\frac{\nu}{2}-\frac{\sqrt{4\mu-\nu^2}}{2}\times\tan\left(\frac{\sqrt{4\mu-\nu^2}}{2}\zeta\right)\right)^2$$

$$+\int f(t)\mathrm{d}t. \tag{34}$$

When $\nu^2 - 4\mu = 0$, we get the rational solution

$$u_6(\zeta) = j_0 - \frac{12c^2\beta\nu}{\alpha}\left(-\frac{\nu}{2} + \frac{Q}{P+Q\zeta}\right) + -\frac{12c^2\beta}{\alpha}\left(-\frac{\nu}{2}\frac{Q}{P+Q\zeta}\right)^2 + \int f(t)\mathrm{d}t. \tag{35}$$

If we put $P = 0$ and $Q \neq 0$, after palliation solution (35) becomes

$$u_{61}(\zeta) = j_0 - \frac{12c^2\beta\nu}{\alpha}\left(-\frac{\nu}{2} + \frac{1}{\zeta}\right) - \frac{12c^2\beta}{\alpha}\left(-\frac{\nu}{2} + \frac{1}{\zeta}\right)^2 + \int f(t)\mathrm{d}t. \tag{36}$$

3.2 *Implementation of the Kudryashov method*

To solve eq. (3), we apply the Kudryashov method, one of the most direct and effective algebraic techniques. Balancing WW' and W''', eq. (9) has the form

$$W(\zeta) = d_0 + d_1 Z(\zeta) + d_2(Z(\zeta))^2. \tag{37}$$

Substituting eqs (37) and (10) into (19) and equating the coefficients for $(Z(\zeta))^i$ (where $i = 1, 2, 3, 4, 5$) to zero, a set of algebraic equations is obtained. Solving the system of equations using Maple provides the following solution:

$$c = c, \quad d_0 = d_0, \quad d_1 = \frac{12c^2\beta}{\alpha}, \quad d_2 = \frac{-12c^2\beta}{\alpha},$$
$$\rho(t) = -c^3\beta - c\alpha\int f(t)\,\mathrm{d}t - \alpha c d_0. \tag{38}$$

Utilising (38) in (37), we get

$$W(\zeta) = d_0 + \frac{12c^2\beta}{\alpha}Z(\zeta) - \frac{12c^2\beta}{\alpha}(Z(\zeta))^2. \tag{39}$$

Hence, eq. (16) implies

$$u(\zeta) = d_0 + \frac{12c^2\beta}{\alpha}Z(\zeta) - \frac{12c^2\beta}{\alpha}(Z(\zeta))^2 + \int f(t)\,\mathrm{d}t. \tag{40}$$

Putting the value of $Z(\zeta)$ from (11) into (40), we finally achieve

$$u(\zeta) = d_0 + \frac{12c^2\beta}{\alpha} \times \frac{\mathrm{e}^{\zeta}}{(1+\mathrm{e}^{\zeta})^2} + \int f(t)\,\mathrm{d}t, \tag{41}$$

where

$$\zeta = cx + \int\left(-c^3\beta - c\alpha\int f(t)\mathrm{d}t - \alpha c d_0\right)\mathrm{d}t.$$

3.3 *Implementation of the $(1/G')$-expansion method*

By balancing WW' and W''', the solution takes the following form:

$$W(\zeta) = \alpha_0 + \alpha_1(1/G') + \alpha_2(1/G')^2. \tag{42}$$

If we proceed like the previous methods, we obtain a set of five algebraic equations. Solving those equations by Maple, we get the following solution:

$$c = c, \quad \alpha_0 = \alpha_0,$$
$$\alpha_1 = -\frac{12c^2\beta P_1 P_2}{\alpha}, \tag{43}$$
$$\alpha_2 = -\frac{12\beta c^2 P_2^2}{\alpha},$$
$$\rho(t) = -\beta c^3 P_1^2 - c\alpha\int f(t)\mathrm{d}t - \alpha c\alpha_0.$$

Substituting (43) into (42), we get

$$W(\zeta) = \alpha_0 - \frac{12c^2\beta P_1 P_2}{\alpha}\left(\frac{1}{G'(\zeta)}\right) - \frac{12\beta c^2 P_2^2}{\alpha}\left(\frac{1}{G'(\zeta)}\right)^2. \tag{44}$$

So eq. (12) becomes

$$u(\zeta) = \alpha_0 - \frac{12c^2\beta P_1 P_2}{\alpha}\left(\frac{1}{G'(\zeta)}\right) - \frac{12\beta c^2 P_2^2}{\alpha}\left(\frac{1}{G'(\zeta)}\right)^2 + \int f(t)\mathrm{d}t. \tag{45}$$

By using the values of $1/G'(\xi)$ from (15), we finally obtain

$$u(\zeta) = \alpha_0 - \frac{12c^2\beta P_1 P_2}{\alpha} \times \left(\frac{P_1}{-P_2 + P_1 s_1(\cosh(\xi P_1) - \sinh(\xi P_1))}\right) - \frac{12\beta c^2 P_2^2}{\alpha} \times \left(\frac{P_1}{-P_2 + P_1 s_1(\cosh(\xi P_1) - \sinh(\xi P_1))}\right)^2 + \int f(t)\mathrm{d}t, \tag{46}$$

where

$$\zeta = cx + \int\left(-\beta c^3 P_1^2 - c\alpha\int f(t)\mathrm{d}t - \alpha c\alpha_0\right)\mathrm{d}t.$$

4. Results and discussion

In this section, we will examine the effects of external time-dependent force on the obtained solutions for

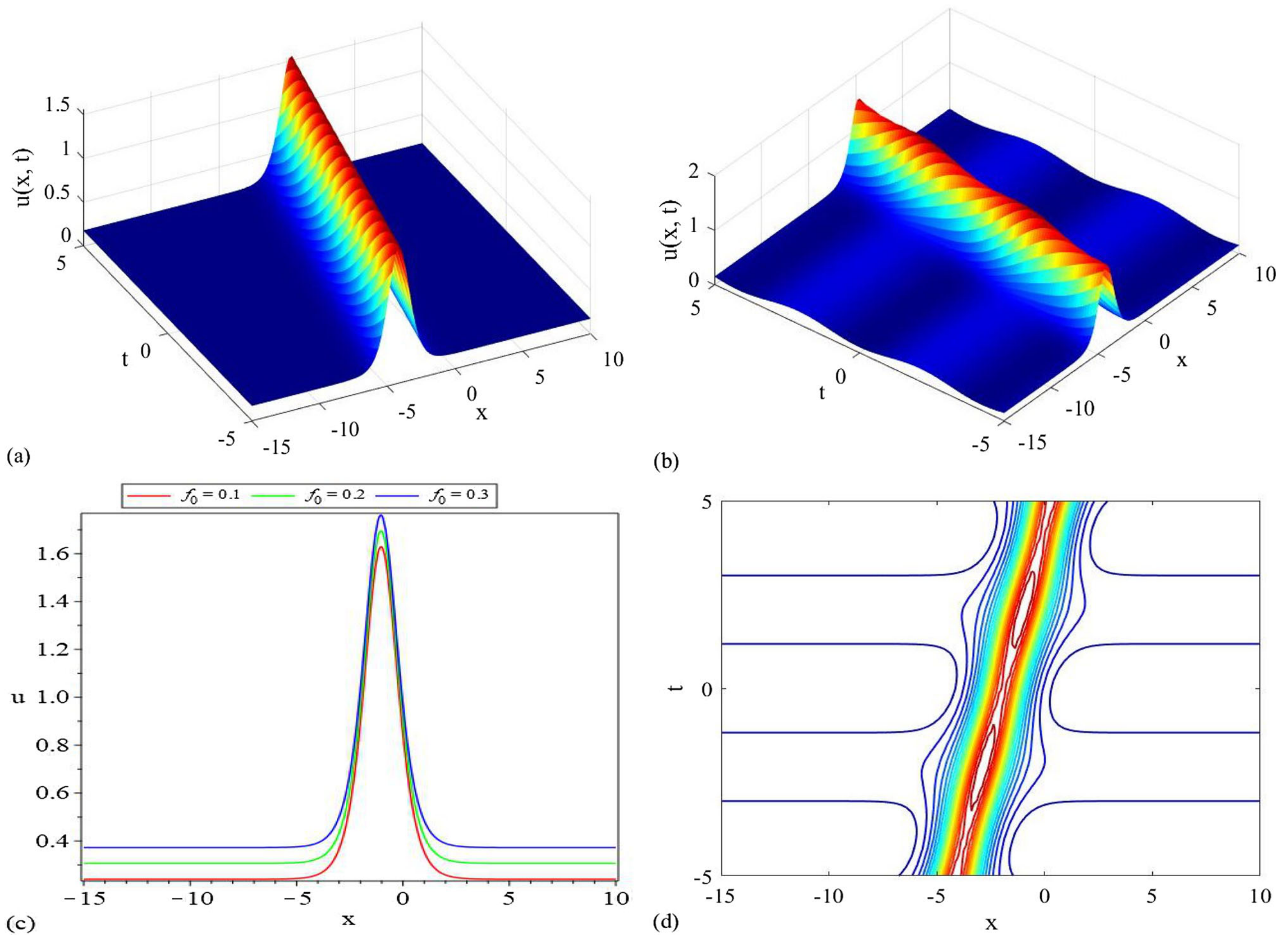

Figure 1. The 3D graphs of solitary wave solution (25), by considering $c = -1$, $\nu = 2$, $\mu = 0.1$, $\alpha = 0.7$, $\beta = 0.09$, $j_0 = 0.02$, $f(t) = f_0 \sin(\omega t)$ and (**a**) for $f_0 = 0$, (**b**) for $f_0 = 0.1$, $\omega = 1.5$. (**c**) The 2D graph with different f_0 corresponding to figure 1b when $t = 2$ and (**d**) is the contour plot of figure 1b.

different force functions with the assistance of 3D, 2D and contour plots. We consider three distinct external forces to explain their impact on the solitary wave solutions, which are derived using three different methods. For solution (25), we investigate the periodic force $f_0 \sin(\omega t)$, in which f_0 and ω represent the strength and frequency of the force, respectively. To explain solutions (41) and (46), we employ the Gaussian-shape force ($g_0 \mathrm{e}^{-\lambda^2 t^2}$) and hyperbolic force ($h_0 \mathrm{sech}^2(\delta t)$), respectively. In these cases, the parameters g_0 and h_0 control the amplitude, while λ and δ regulate the width of the solutions.

Figure 1 illustrates the effect of periodic forcing ($f(t) = f_0 \sin(\omega t)$) on the solitary wave solution (25) with the parameters set as follows: $c = -1$, $\nu = 2$, $\mu = 0.1$, $\alpha = 0.7$, $\beta = 0.09$, $j_0 = 0.02$, $f_0 = 0.1$ and $\omega = 1.5$. The 3D plot in the absence of forcing ($f_0 = 0$) is depicted in figure 1a. The overall impact of the forcing component (f_0) is presented in figure 1b. In this case, increasing the forcing leads to the creation of periodic surroundings and elevates the background of the solitary wave. The background of the solitary wave increases as the magnitude of f_0 increases, as evident in figure 1c. To gain a deeper understanding of the primary influence of periodic forcing, the relevant contour of figure 1b is displayed in figure 1d. Following the same procedure, it can be shown that as the magnitude of ω gradually increases, the background of the solitary wave also rises and beyond certain values of ω, it exhibits periodic behaviour within a range of u.

Figure 2 represents the effect of a Gaussian-shaped external force, given by $f(t) = g_0 \mathrm{e}^{-\lambda^2 t^2}$, on the solitary wave solution (41) for specific parameter values: $c = 1$, $\alpha = 0.5$, $\beta = 0.3$, $d_0 = 0.1$, $g_0 = 0.1$ and $\lambda = 1$. In figure 2a, we observe the 3D plot in the absence of external forcing ($g_0 = 0$). The impact of the forcing component (g_0) becomes evident in figure 2b. As we increase the magnitude of the forcing, we notice the emergence of soliton that spreads across a kink-type canvas. Furthermore, in figure 2c, we observe that the background of

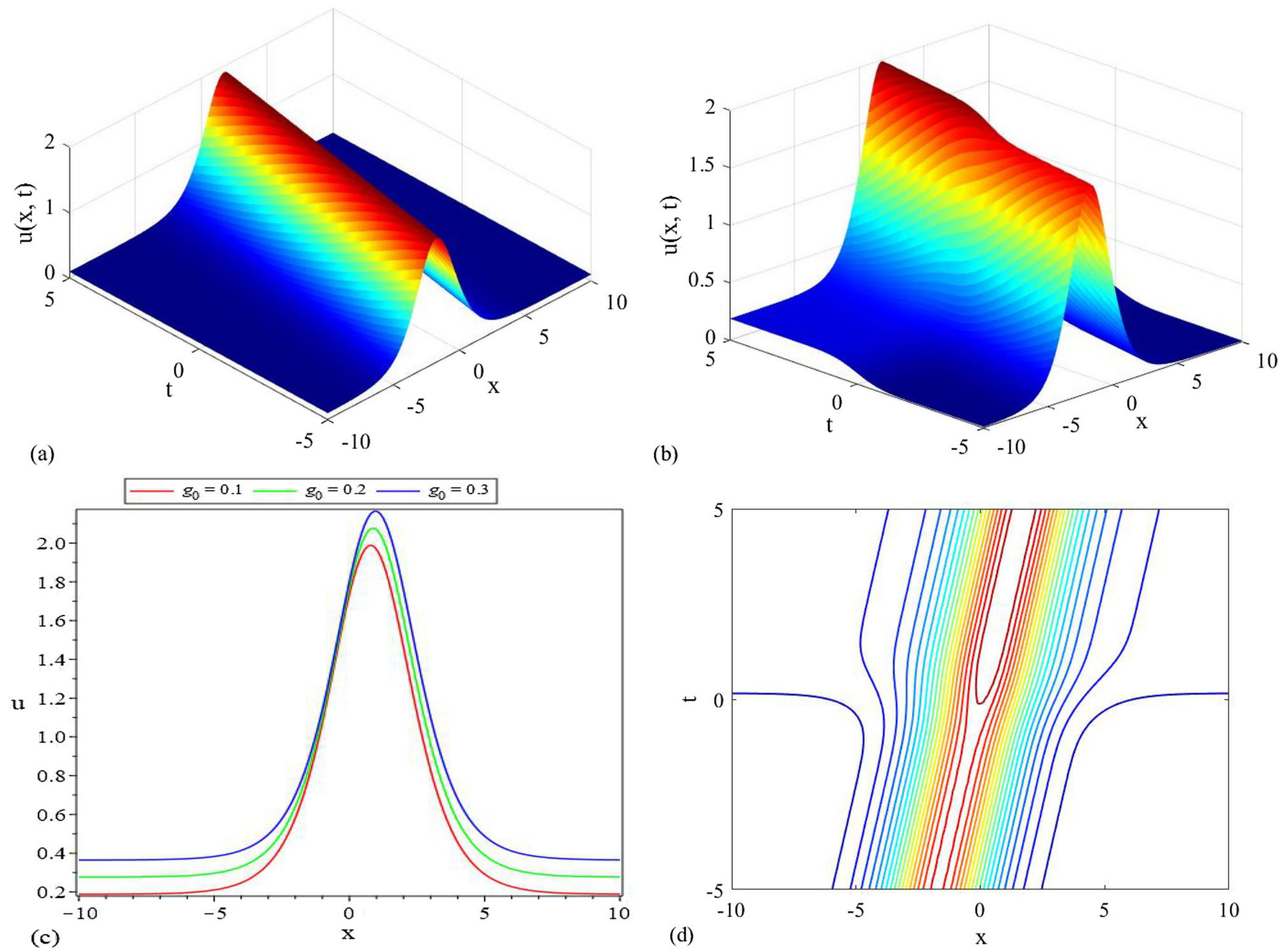

Figure 2. The 3D graphs of solitary wave solution (41), by considering $c = 1$, $\alpha = 0.5$, $\beta = 0.3$, $d_0 = 0.1$, $f(t) = g_0 e^{-\lambda^2 t^2}$ and (**a**) for $g_0 = 0$, (**b**) for $g_0 = 0.1$, $\lambda = 1$. (**c**) The 2D graph with different g_0 corresponding to figure 2b when $t = 2$ and (**d**) the contour plot of figure 2.

the solitary wave rises as the magnitude of g_0 increases. To further investigate the influence of the variable ω, we can consider g_0 as a positive constant within the force expression $f(t) = g_0 e^{-\lambda^2 t^2}$ using the same approach. In order to provide a clearer picture of the main impact of periodic forcing, figure 2d presents the corresponding contour of figure 2b. In a similar procedure, it is noteworthy that as the absolute value of λ increases while keeping g_0 fixed, the background of the solitary wave gradually decreases.

Figure 3 demonstrates the influence of a hyperbolic external force ($f(t) = h_0 \mathrm{sech}^2(\delta t)$) on the solitary wave solution given by eq. (46) when certain parameters are fixed: $c = 1$, $P_1 = 1$, $P_2 = 2$, $s_1 = -1$, $\alpha = 0.5$, $\beta = 1$, $\alpha_0 = 0.2$, $h_0 = 0.3$ and $\delta = 1$. In figure 3a, we can observe the 3D plot when no external force is applied ($h_0 = 0$). Figure 3b illustrates the overall impact of the external forcing component (h_0). Here, an increase in the forcing leads to the emergence of a soliton with a kink-shaped background. Figure 3c demonstrates how the background and speed of the solitary wave increase as the magnitude of h_0 is raised. Figure 3d shows the corresponding contour of figure 3b to provide a clearer understanding of the main impact of periodic forcing. In a similar process, it is interesting to note that the background of the solitary wave gradually diminishes as the absolute value of δ rises while maintaining h_0 fixed.

Mathematically, in contrast to Zhang [49], who used the exponential function approach to find solutions to the fKdV equation, we used the Kudryashov method to get the exponential function solutions. We have found that the obtained solutions are of different forms [49]. Manafian and Nasrollahpour [62] used the (G'/G)-expansion approach to solve the forced KdV equation and the results are given in the form $u(\zeta) = a_0 + a_1(G'/G) + a_2(G'/G)^2$, where $G = G(\zeta)$ satisfies the differential eq. (7). In comparison to [62], we have obtained the solutions for the fKdV equation by utilising the extended (G'/G)-expansion approach as $u(\zeta) = a_0 + a_1(G'/G) + a_2(G'/G)^2 + b_1(G'/G)^{-1} +$

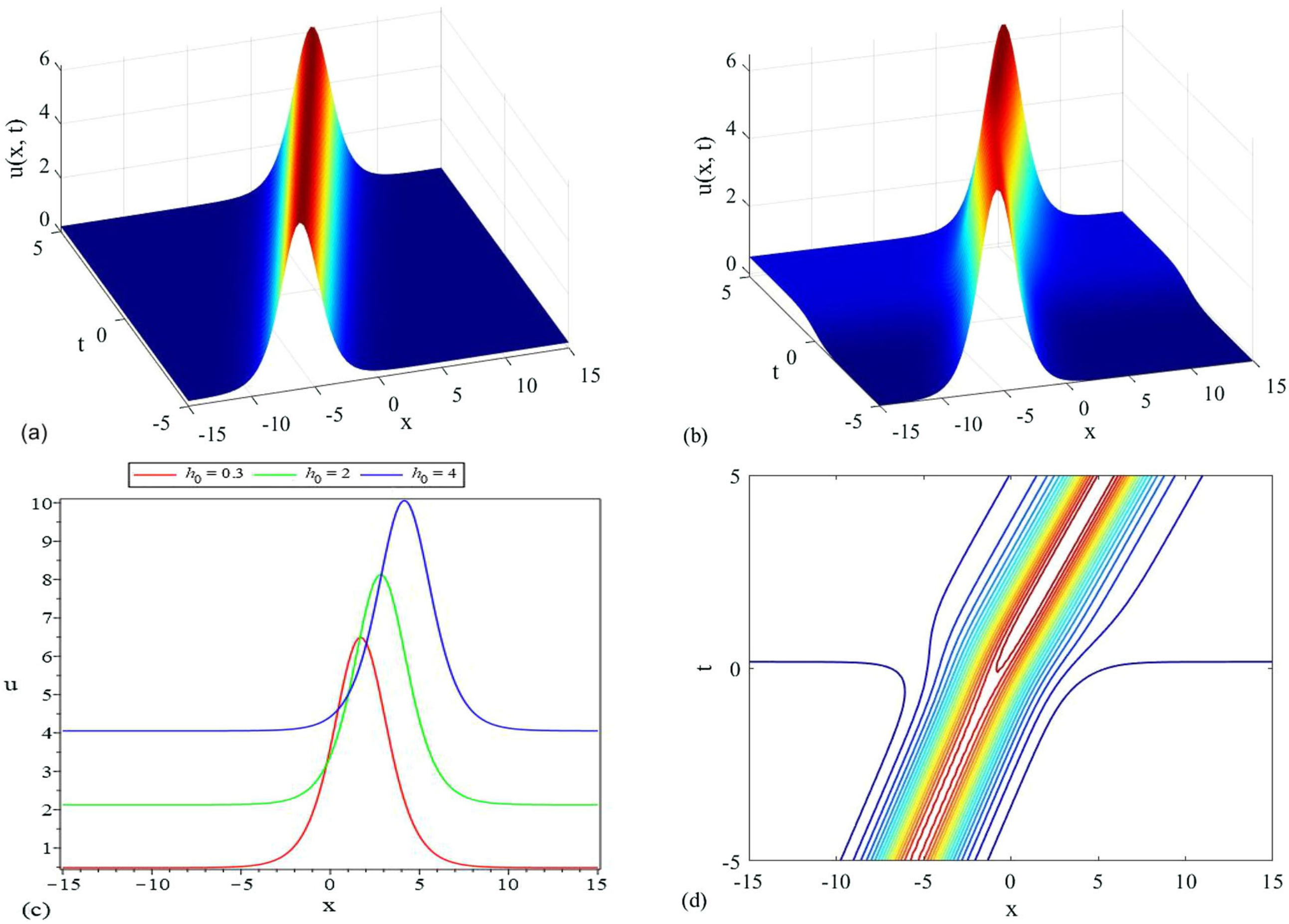

Figure 3. The 3D graphs of solitary wave solution (46) by considering $c = 1$, $P_1 = 1$, $P_2 = 2$, $s_1 = -1$, $\alpha = 0.5$, $\beta = 1$, $\alpha_0 = 0.2$ and (**a**) for $h_0 = 0$, (**b**) for $h_0 = 0.3$, $\delta = 1$. (**c**) The 2D graph with different h_0 corresponding to figure 3b when $t = 2$ and (**d**) the contour plot of figure 3b.

$b_2(G'/G)^{-2}$, where $G = G(\zeta)$ satisfies the differential eq. (7). We have more generalised solutions than Manafian and Nasrollahpour [62], because in our case, if both b_1 and b_2 are zero, then the obtained solutions are similar to those solutions in [62], but also some of our other solutions are entirely different from [62].

5. Conclusions

The solutions for the fKdV equation have been derived successfully by using the extended (G'/G) expansion technique, the Kudryashov method and the $(1/G')$ expansion methods. The obtained solutions are in trigonometric, hyperbolic, exponential and rational functions form. These wave solutions have been verified by putting them back into the original equation. Most of the solutions obtained are completely new compared to previous literature. We can clearly observe that these approaches are robust and effective tools for solving a variety of NLEEs. The graphical representation of the obtained solutions are furnished through the 3D, 2D and contour plots by providing suitable values of free parameters. The external force has a significant impact on the formation of soliton backgrounds. For instance, when a trigonometric forcing component is applied, periodicity increases as f_0 increases. Again, a kink-type backdrop emerges as a result of the hyperbolic and Gaussian-shaped forcing components. Figures 1–3 demonstrate how the external time-dependent forces affect the solitary waves. It can be observed from the graphical discussion that the external forces affect the background and speed of solitary waves. The obtained results are expected to be valuable in addressing the propagation of solitary waves subjected to external forces, which can be described by eq. (3).

It is true that eq. (3) is more realistic when the external force depends on both position (x) and time (t), but obtaining reliable information analytically in such cases is challenging. That is the reason why we have studied the case of x-independent external force. Furthermore, the authors believe that the fKdV model (3) may still offer useful information as a good approximation when the characteristic length of the soliton is much smaller than the coherence length of the external force. In future, the extended (G'/G)-expansion method, the Kudrayashov method and the $(1/G')$-expansion method can be employed to investigate exact solutions for NLEEs with variable coefficients and force terms. These equations have numerous applications in the fields of science and engineering. Additionally, we can also study the more general forced KdV equation when the force function depends on both x and t. Maple and Matlab were used to perform all the symbolic computations and graphical work.

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